A Child's Guide to Log Linearization

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1 What are we trying to do?

What do macroeconomists mean when they say "log-linearize an equation around steady state"? They mean, rewrite each variable x in the equation as $\log\left(\frac{x}{\bar{x}}\right)$, where $\bar{x}$ is the steady state value of x. Why do we want to do this? Two reasons:

1.1 Log converts multiplicative relationships to additive relationships

The equation will now be linear! It's inconvenient (for doing regressions, writing code) when the relationship between the variables is nonlinear. log is a function that has two very special properties: $\log(ab) = \log a + \log b$ and $\log\left(a^b\right) = b \log a$. log can convert multiplication to addition, and exponents to multiplication. Why is this relevant to linearization? For example, suppose $f(x) = x^2$, a quadratic. Why don't we instead express x in terms of its log? We can rewrite as $\log f(x) = 2 \log x$. Define $y = \log x$. Then we can rewrite the equation as $f(y) = 2y$. We just went from a quadratic to a linear equation!

1.2 Log allows for percentage interpretations

log allows us to interpret changes in variables as changes in their percentages. This is especially useful when we analyze variables with very different magnitudes. As economists we love thinking about things in terms of deviations and percentages, rather than the levels, because percentages allow us to compare apples to oranges: for example, if one industry much more sales than another industry, their levels of sales may not be comparable, but the percentage change in sales will be. This is especially useful when we want to understand and evaluate the effects of different policies. Percentages are a natural way to think of the response of some economic variable to a policy impact, and allows us to compare the responses of diverse parts of the economy that may not otherwise be comparable in levels (ex. employment, sales, prices, are often very different in their levels).

Specifically, in macro we like to talk about deviations from the steady state. The steady state is defined as the values for which the model's dynamic variables will not change from one period to the next. (The steady state is almost never attained in real life: it's a convenient mathematical entity. But solving for it certainly helps us think about transitions from one steady state to another. Arguably, the economy is always "transitioning", so that does correspond more to real life.) Now log brings us from the realm of levels to percentage deviations from a steady state. Why?

Suppose we are initially at $\bar{x}$ and we deviate to x. In logs, the deviation would be $\log x - \log \bar{x} = \log \frac{x}{\bar{x}}$. For small deviations of x, we can approximate $\log x$ with the Taylor rule. The first order approximation of $\log \frac{x}{\bar{x}}$ around $x = \bar{x}$ is

$$\begin{aligned} \log x - \log \bar{x} &= \log \frac{x}{\bar{x}}|_{x=\bar{x}} \\ &\approx \log(1) + \frac{1}{\bar{x}/\bar{x}} \frac{1}{\bar{x}} (x - \bar{x}) = \frac{x - \bar{x}}{\bar{x}} \end{aligned}$$

Therefore, remarkably, $\log x - \log \bar{x} \approx \frac{x-\bar{x}}{\bar{x}}$. Thus for any variable x, we can approximate percentage deviations from some arbitrary level $\bar{x}$ with deviations of $\log x$ from $\log \bar{x}$. This fact helps us interpret the results of regressing y against x, when y and x are on vastly different scales: for example, we could have $y \in [0, 20]$ but $x \in [2000, 4000]$. Suppose we want to recover estimates $\hat{\alpha}, \hat{\beta}$ where we believe the relationship between x and y to be $y = \hat{\alpha} + \hat{\beta} x$. Even if x strong predicts y, the coefficient $\hat{\beta}$ may be very small, since x tends to be large (between 2000 and 4000) and y tends to be small (between 0 and 20). It is much more informative to look at the response of y to x when x changes by some percentage, the elasticity so to speak.

Let us denote the log deviation around steady state x as $\hat{x} \equiv \log \frac{x}{\bar{x}}$.

2 A Simple Example from First Principles

Suppose we wanted to log linearize

$$y_t = s z_t k_t^{\alpha}$$

Notice that the steady state $\bar{y}$ is simply $\bar{y} = s\bar{z}\bar{k}^{\alpha}$. Dividing by steady state values and taking logs, we have

$$\begin{aligned} \frac{y_t}{\bar{y}} &= \frac{z_t}{\bar{z}} \left(\frac{k_t}{\bar{k}} \right)^{\alpha} \\ \log \frac{y_t}{\bar{y}} &= \log \frac{z_t}{\bar{z}} + \alpha \log \frac{k_t}{\bar{k}} \\ \hat{y}_t &= \hat{z}_t + \alpha \hat{k}_t \end{aligned}$$

3 Cookbook Procedure

For ease of notation denote $dx_t \equiv x_t - \bar{x}$. Suppose we wanted to log linearize an arbitrary equation

$$f(x_t, w_t) = g(y_t, z_t)$$

It does not matter the number of variables in f and g!

Step 1.

Take the partial derivatives, where we tack on to the partial derivative term of x_t the term dx_t. Yes, right now it is completely arbitrary. But see "Proof of the Cookbook Procedure" to learn why!

Evaluate the partial derivatives at steady state values $\bar{x}, \bar{w}, \bar{y}, \bar{z}$.

$$\frac{\partial}{\partial x_t} f(\bar{x}, \bar{w})\, dx_t + \frac{\partial}{\partial w_t} f(\bar{x}, \bar{w})\, dw_t = \frac{\partial}{\partial y_t} g(\bar{y}, \bar{z})\, dy_t + \frac{\partial}{\partial z_t} g(\bar{y}, \bar{z})\, dz_t$$

Step 2.

For each dx_t, substitute in the term $\hat{x}_t \bar{x}$, where $\hat{x}_t = \log\left(\frac{x_t}{\bar{x}}\right)$. Simplify and you are done!

$$f_1(\bar{x}, \bar{w})\, \hat{x}_t \bar{x} + f_2(\bar{x}, \bar{w})\, \hat{w}_t \bar{w} = g_1(\bar{y}, \bar{z})\, \hat{y}_t \bar{y} + g_2(\bar{y}, \bar{z})\, \hat{z}_t \bar{z}$$

Indeed, in the proof below I will show you that $dx_t \equiv x_t - \bar{x} \approx \hat{x}_t \bar{x}$.

4 More General Example

Suppose we want to log linearize

$$c_t + k_t = w_t n_t$$

Step 1.

Take the partial derivatives evaluated at the steady state, where we tack on to the partial derivative term of x the term dx. Note that the partial derivative $\frac{\partial}{\partial x}(x + \text{anything}) = 1$. In a different case, if the partial still contained x, we would need to sub in for steady state value $\bar{x}$.

$$(1)\, dc_t + (1)\, dk_t = (1)\, \bar{n} dw_t + (1)\, \bar{w} dn_t$$

Step 2.

Substitute in for each arbitrary dx_t the term $\hat{x}_t \bar{x}$.

$$\begin{aligned} \hat{c}_t \bar{c} + \hat{k}_t \bar{k} &= \bar{n} \hat{w}_t \bar{w} + \bar{w} \hat{n}_t \bar{n} \\ \hat{c}_t \bar{c} + \hat{k}_t \bar{k} &= \bar{w} \bar{n} \left(\hat{w}_t + \hat{n}_t \right) \end{aligned}$$

And that's it, folks!

5 Proof of the Cookbook Procedure

Suppose we wanted to log linearize an arbitrary equation

$$f(x) = g(y, z)$$

It does not matter the number of variables in f and g! What I show below will still work.

Take the Taylor expansion around the steady state of both sides of the equation, where $\bar{x}$, $\bar{y}$, and $\bar{z}$ are the steady state values:

$$f(\bar{x}) + f'(\bar{x})(x - \bar{x}) = g(\bar{y}, \bar{z}) + \frac{\partial}{\partial y} g(\bar{y}, \bar{z})(y - \bar{y}) + \frac{\partial}{\partial z} g(\bar{y}, \bar{z})(z - \bar{z}) \qquad (1)$$

The right hand side is the standard multinomial Taylor expansion.

Now importantly, note that this will always be true, almost tautologically:

$$f(\bar{x}) = g(\bar{y}, \bar{z})$$

Remember, in economics we only consider unique steady state values, and in order to be considered a steady state ALL the dynamic variables have to be in steady state.

We can then just eliminate the constant term, so that equation 1 becomes

$$f'(\bar{x})(x-\bar{x}) = \frac{\partial}{\partial y} g(\bar{y},\bar{z})(y-\bar{y}) + \frac{\partial}{\partial z} g(\bar{y},\bar{z})(z-\bar{z})$$

Now recall that the first order Taylor approximation around $\bar{x}$ of $\log\left(\frac{x}{\bar{x}}\right)|_{x=\bar{x}} \approx \frac{x-\bar{x}}{\bar{x}}$. Therefore

$$\begin{aligned} x-\bar{x} &\approx \log\left(\frac{x}{\bar{x}}\right)\bar{x} \\ &\equiv \hat{x}\bar{x} \end{aligned}$$

Let's sub in for all the $x-\bar{x}$ in equation 1 to get

$$f'(\bar{x})\hat{x}\bar{x} = \frac{\partial}{\partial y} g(\bar{y},\bar{z})\hat{y}\bar{y} + \frac{\partial}{\partial z} g(\bar{y},\bar{z})\hat{z}\bar{z}$$

This is exactly what we did above! As an exercise, you can show this where f is a function of two variables, for example, x and w. But hopefully it is clear that the proof is independent of the number of variables.

6 More Complex Examples

6.1 Resource Constraint

Suppose you want to log-linearize

$$A_t k_t^{\alpha} = c_t + k_{t+1} - (1-\delta) k_t$$

Step 1.

Take the partial derivatives evaluated at the steady state, where we tack on to the partial derivative term of x the term dx.

$$\bar{k}^{\alpha} dA_t + \bar{A}\alpha\bar{k}^{\alpha-1} dk_t = dc_t + dk_{t+1} - (1-\delta) dk_t$$

Step 2.

Substitute in $dx_t = \hat{x}_t\bar{x}$.

$$\begin{aligned} \bar{k}^{\alpha}\hat{A}_t\bar{A} + \bar{A}\alpha\bar{k}^{\alpha-1}\hat{k}_t\bar{k} &= \hat{c}_t\bar{c} + \hat{k}_{t+1}\bar{k} - (1-\delta)\hat{k}_t\bar{k} \\ \bar{A}\bar{k}^{\alpha}\left(\hat{A}_t + \alpha\hat{k}_t\right) &= \hat{c}_t\bar{c} + \bar{k}\left(\hat{k}_{t+1} - (1-\delta)\hat{k}_t\right) \end{aligned}$$

6.2 Euler Equation

Suppose you want to log-linearize

$$c_t^{-\sigma} = \beta \left(\alpha A_t k_{t+1}^{\alpha-1} + 1 - \delta \right) c_{t+1}^{-\sigma}$$

Step 1.

Take the partial derivatives evaluated at the steady state, where we tack on to the partial derivative term of x the term dx.

$$-\sigma \bar{c}^{-\sigma-1} dc_t = \beta \left(\alpha \bar{k}^{\alpha-1} dA_t + \alpha \bar{A} \left(\alpha - 1\right) \bar{k}^{\alpha-2} dk_{t+1} \right) \bar{c}^{-\sigma} + \beta \left(\alpha \bar{A} \bar{k}^{\alpha-1} + 1 - \delta \right) \left(-\sigma \bar{c}^{-\sigma-1} dc_{t+1} \right)$$

Step 2.

Substitute in $dx_t = \hat{x}_t \bar{x}$.

$$\begin{aligned}
-\sigma \bar{c}^{-\sigma-1} \hat{c}_t \bar{c} &= \beta \left(\alpha \bar{k}^{\alpha-1} \hat{A}_t \bar{A} + \alpha \bar{A} \left(\alpha - 1\right) \bar{k}^{\alpha-2} \hat{k}_t \bar{k} \right) \bar{c}^{-\sigma} + \beta \left(\alpha \bar{A} \bar{k}^{\alpha-1} + 1 - \delta \right) \left(-\sigma \bar{c}^{-\sigma-1} \hat{c}_t \bar{c} \right) \\
-\sigma \bar{c}^{-\sigma} \hat{c}_t &= \beta \bar{c}^{-\sigma} \left\{ \alpha \bar{A} \bar{k}^{\alpha-1} \left(\hat{A}_t + \left(\alpha - 1\right) \hat{k}_t \right) - \sigma \left(\alpha \bar{A} \bar{k}^{\alpha-1} + 1 - \delta \right) \hat{c}_t \right\} \\
-\sigma \hat{c}_t &= \beta \left\{ \alpha \bar{A} \bar{k}^{\alpha-1} \left(\hat{A}_t + \left(\alpha - 1\right) \hat{k}_t \right) - \sigma \left(\alpha \bar{A} \bar{k}^{\alpha-1} + 1 - \delta \right) \hat{c}_t \right\}
\end{aligned}$$