Musings on Macro

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1 A Real Business Cycle Model

We will construct a dynamic model of the economy. The RBC model is the workhorse for countless more advanced macro models, where more bells and whistles are added to more closely reflect our reality. The pool of macro models that hail the RBC model as their common ancestor are aptly called dynamic stochastic general equilibrium "DSGE" models.

Why dynamic? It's not useful to consider the economy at a specific point in time, because we often want to make predictions about the future or explain why one event caused another.

Why stochastic? We live in a fundamentally random world (just ask the quantum physicists!). We can get into a metaphysical or scientific discussion about whether we can predict the future if we knew the movement of every particle in the universe. However, currently it is impossible to know that, so to deal with our imperfect knowledge we assume certain things proceed according to a probability distribution. Of course, all these random events are indexed by time, hence "stochastic".

Why general equilibrium? We care about broad, aggregate effects in the long run. General equilibrium is a state of the model when "all markets clear"; that is, when supply equals demand and all prices are endogenously determined. In the interim (the transition path), prices will adjust, and this movement of prices will continue to influence the economy. Refrain here from arguing "the market never reaches general equilibrium": remember, we're not claiming it does, but rather that solving for the equilibrium state is a useful anchor for further analysis.

An indispensable tool to solve the RBC model (and thus DSGE models) is "dynamic programming".

An important framework we use is rational expectations. This does not imply that we think people are always "rational", whatever that means, but rather that we assume our agents are acting according to mathematical expectations, according to their available information. This framework is useful, because we can access all the tools of probability and statistics.

1.1 History

Arrow and Debreu were pioneers of general equilibrium theory. The motivation behind these DSGE models include as well the first and second welfare theorems.

First Welfare Theorem: If people are competing with each other and prices are allowed to adjust, you can't make one person better off without making someone else worse off.

Second Welfare Theorem: Suppose a benevolent omnipotent dictator were allowed to choose the resource allocation for everybody. We can achieve this allocation by allowing prices adjust and making transfers between people.

The welfare theorems are a good reason why economists tend to like capitalism: we want to find a Pareto optimum, and that's hard! Usually our preferences exclude envy, although this may not be a useful assumption.

1.2 The social planner problem

Given our available capital k_{-1}, initial technology A_0, our "value" is given by

$$\begin{aligned} V_0 &= \max_{c_t, n_t, k_t, y_t} E_0\left[\sum_{t=0}^{\infty} \beta^t u\left(c_t, n_t\right)\right] \\ \text{s.t. } c_t + k_t &= y_t + (1-\delta)\, k_{t-1} \\ y_t &= f\left(A_t \frac{n_t}{k_{t-1}}\right) k_{t-1} \end{aligned}$$

and some exogenous process for $A_t > 0$. For example,

$$\log A_t = \rho \log A_{t-1} + \varepsilon_t, \ \ \varepsilon_t \overset{iid}{\sim} N\left(0, \sigma^2\right)$$

The value function is a manufactured concept, which aggregates our utility at every point in time, subject to some resource constraints. We always assume in macro that we are impatient: we prefer to enjoy things now rather than later. This is clearly not a crazy assumption about human nature, and mortality.

In the formulation, we are agnostic about functional form of our utility, but we assume that it is only dependent on two things: how much we consume and how much we work. We make another non-crazy assumption about human nature: we like to have things but we don't like to work. The resource constraint says, we are allowed to consume c_t and save k_t. Our savings depreciate over time by a constant amount, according to $1 - \delta$. We get a wage every period y_t, and hence the total available to spend each period is $y_t + (1-\delta)\, k_{t-1}$. Our wage is given by our output $y_t = f\left(A_t \frac{n_t}{k_{t-1}}, k_{t-1}\right)$. A_t can be interpreted as labour augmenting technology at time t.

Note that with Cobb-Douglas assumptions with a labour share of α, this is a constant returns to scale CRS production where $y_t = \left(A_t \frac{n_t}{k_{t-1}}\right)^{\alpha} k_{t-1}$.

Remark: in this model, the social planner values every household the same. Obviously, we can do something different, where the planner favours some households over others, and we would still be in a Pareto optimum.

Remark: There is a large set of possible preferences macroeconomists use for $u\left(c_t, n_t\right)$. You many expect that the preferences we choose can have large consequences for the predictions of our models, and that is correct! Which preferences we ought to use in our models is the subject of oft-discussed macro papers and fiercely debated. Unfortunately, it is difficult to confirm that any particular preference is correct, due to how we cannot directly observe preference in the data - but that hasn't stopped people from making inferences.

1.3 Production and profit maximization

Consider the decentralized economy. Profit maximization is defined as, taking wages w_t and capital rental rates r_t as given, the "firm" solves

$$\max_{k_{t-1}, n_t} f\left(A_t \frac{n_t}{k_{t-1}}\right) k_{t-1} - w_t n_t - r_t k_{t-1}$$

The first-order conditions (FOC) are simply the labour and capital market clearing equations (where price of labour and capital and equal to their demands):

$$\begin{aligned} w_t &= f'\left(A_t \frac{n_t}{k_{t-1}}\right) A_t \frac{1}{k_{t-1}} \\ \Longrightarrow w_t n_t &= f'\left(A_t \frac{n_t}{k_{t-1}}\right) A_t \frac{n_t}{k_{t-1}} \\ r_t &= f\left(A_t \frac{n_t}{k_{t-1}}\right) - f'\left(A_t \frac{n_t}{k_{t-1}}\right) A_t \frac{n_t}{k_{t-1}} \\ \Longrightarrow r_t k_{t-1} &= y_t - w_t n_t \end{aligned}$$

Why do we rewrite the FOC's like this? Because we can interpret $w_t n_t$ and $r_t k_{t-1}$ as the labour share and capital share of output, respectively. Why is this interpretation interesting? Well, it probably goes all the way back to the notion of class struggle as put forth by Karl Marx (In fact, I was told that we use K for capital instead of other letters like C because of Marx! Indeed, capital in German is "kapital"). We want to know how much of what we produce is going to labour as opposed to capital, to the working class, who collect wages w, as opposed to the landed class, who collect capital rents r.

In competitive equilibrium, profits are zero, because we assume in this model that each firm is infinitesimally small and cannot individually influence prices by changing their supply.

Remark: Remember, we are not claiming the setup in RBC is "true", or the best setup to describe the economy. Rather, we are making a bunch of very strong assumptions to set up a workhorse to build more complicated models.

1.4 Dynamic Programming

Dynamic programming is a tool introduced by Richard Bellman in the 1950s, originally used for mathematical optimization and computer programming. Why was it useful (and still useful) for computer programming? Because as opposed to framing our optimization problem sequentially, as we do in time series analysis, in dynamic programming we frame our problem recursively. Recursion is very useful for programming, because computers don't care if we tell it to do the same task a million times. Dynamic models have a repetitive nature because at different points in time, we are often solving the same problem. Hence, we can break down an otherwise complicated problem into simpler sub-problems.

1.4.1 A recursive structure

How do we move from the sequential formulation of the problem to the recursive one? Using the law of iterated expectations, we can express our previous problem as:

$$\begin{aligned} V_0 &= E_0\left[\sum_{t=0}^{\infty}\beta^t u\left(c_t, n_t\right)\right] \\ &= u\left(c_0, n_0\right) + \beta E_0\left[E_t\left[\sum_{s=0}^{\infty}\beta^s u\left(c_{s+1}, n_{s+1}\right)\right]\right] \\ &= u\left(c_0, n_0\right) + \beta E_0\left[V_1\right] \end{aligned}$$

Note that we have now written the value function V in terms of itself, but just shifted one period forward in time!

State of the economy: (k, A) where

$$(k, A) = (k_{-1}, A_0) \text{ for } t = 0 \text{ and } (k', A') = (k_0, A_1) \text{ for } t = 1$$

Write V as a function of the state, rather than time:

$$\begin{aligned} V(k, A) &= V_0 \\ &= u\left(c_0, n_0\right) + \beta E_0\left[V\left(k_0, A_1\right)\right] \\ &= u\left(c_0, n_0\right) + \beta E\left[V\left(k', A'\right) | k, A\right] \end{aligned}$$

Indeed, we can now abstract away time, and only consider our decision today and tomorrow, and do this every day of our lives. One can argue that this reflects better how we make decisions in real life anyways: we don't have a social planner thinking about the sum of our utility over our entire lifetime, but rather we personally wake up and think about today versus tomorrow.

We rewrite the social planner problem as a dynamic programming problem,

$$\begin{aligned} V(k, A) &= \max_{c,n,y,k'}\left\{u(c, n) + \beta E\left[V\left(k', A'\right) | k, A\right]\right\} \qquad (1) \\ \text{s.t. } c + k' &= y + (1-\delta) k \\ y &= f\left(A\frac{n}{k}\right) k \\ \log A' &= \rho \log A + \varepsilon,\ \varepsilon \sim N\left(0, \sigma^2\right) \end{aligned}$$

where the goal is to solve for the value function $V(k, A)$ and the decision or policy rules

$$c(k, A), n(k, A), y(k, A), k'(k, A)$$

The name policy rule is suggestive: we want to find the optimal path of consumption, labour, capital, and output with the given circumstances, that we can then use to make policy recommendations.

Now we want to take the FOC. We can substitute in the resource constraint to eliminate y as a choice variable, leaving us with c,n,v, and k'. However, we can also use a Lagrangian to deal with many variables.

1.4.2 A Lagrangian

Lagrangians allow us to maximize over c,n,k' subject to a constraint. Let $V(k, A) = \max_{c,n,k',\lambda} L(k, A, c, n, k', \lambda)$ where

$$L(k, A, c, n, k', \lambda) = u(c, n) + \beta E[V(k', A')] + \lambda\left(f\left(A\frac{n}{k}\right)k + (1-\delta)k - c - k'\right)$$

Why does it make sense that $V(k, A) = \max_{c,n,k',\lambda} L(k, A, c, n, k', \lambda)$? Here is the ingenuity of this formulation: V is defined as a max in terms of itself (there is a next period V in the Lagrangian L). Indeed, the solution to the Lagrangian L is just V. To convince yourself, look at equation 1 again.

The Lagrangian works as before in the sequential case, where we still write what we are maximizing and add the constraint with a multiplier λ.

The first order necessary conditions or FONC's are:

$$\begin{aligned}
\frac{\partial L}{\partial n} &: \quad 0 = u_n(c, n) + \lambda f'\left(A\frac{n}{k}\right)A\frac{1}{k} \\
\frac{\partial L}{\partial c} &: \quad 0 = u_c(c, n) - \lambda \\
\frac{\partial L}{\partial k'} &: \quad 0 = \beta E[V_1(k', A')] - \lambda \\
\frac{\partial L}{\partial \lambda} &: \quad 0 = f\left(A\frac{n}{k}\right)k + (1-\delta)k - c - k'
\end{aligned}$$

Note that in the capital market clearing equation $\frac{\partial L}{\partial k'}$, we have $V_1(k', A')$. What do we do with that? Continue reading.

NOTE: I use the notation V_1 to refer to the partial differentiation of V by the first argument k'. Why do I not write $V_k(k', A')$? Because this may easily be confused to mean differentiation by k, where k is a function of k'. Indeed, we really ought to write $V_{k'}(k', A')$ but that is clunky and involves writing too many primes.

1.4.3 The Envelope Condition

The purpose here is to find $V_1(k', A')$. Notice that the "income" side of the resource constraint depends on k but not k'. Indeed, it depends completely upon the state variables today. Let's rewrite the income side as $X(k, A)$ in the Lagrangian. We can interpret $X(k, A)$ as our total wealth:

$$\begin{aligned} V(k,A) &= \max_{c,n,k',\lambda} u(c,n) + \beta E[V(k',A')] + \lambda\left(f\left(A\frac{n}{k}\right)k + (1-\delta)k - c - k'\right) \\ &= \max_{c,n,k',\lambda} u(c,n) + \beta E[V(k',A')] + \lambda(X(k,A) - c - k') \end{aligned}$$

Taking the derivative with respect to k,

$$V_k(k,A) = \lambda X_k(k,A) \tag{2}$$

We just used the Envelope Condition by ignoring the fact that k' is a function of k! Indeed, we may naively think we need to use the chain rule, but in the capital FOC $\frac{\partial L}{\partial k'}$, we have already optimized for k', and we know that it must satisfy

$$\lambda = \beta E[V_1(k', A')]$$

Hence, as long as we ultimately make sure this condition holds, we do not need to use the chain rule. Think of the k' as really the $\arg\max k'$.

1.4.4 Indifference at the Optimum

Using the approximation $X_k(k, A) \approx \Delta X/\Delta k$, we write

$$V_k(k,A)\,\Delta k = \lambda \Delta X \tag{3}$$

From the consumption and capital FOCs, we know that

$$\lambda = u_c(c, n)$$

so plugging in,

$$V_k(k, A)\,\Delta k = u_c(c, n)\,\Delta X$$

The right hand side can be rewritten as

$$u_c(c, n)\,\Delta X = u(c + \Delta X, n) - u(c, n)$$

or the marginal utility from consuming our extra wealth ΔX.

The left hand side can be rewritten as

$$V_k(k, A)\,\Delta k = V(k + \Delta k, A) - V(k, A)$$

or the marginal value from an extra unit of capital.

At the optimum, we must be indifferent between consuming our wealth versus getting additional capital.

1.4.5 The Rate of Return

Let's define R to be the rate of return to capital, where

$$R(k, A) \equiv X_k(k, A)$$

and X is our gross investment. (Note that in our model above, our income is purely from gross investment, as opposed to wages).

Recall from the capital FOC that

$$\lambda = \beta E\left[V_1(k', A')\right]$$

If we iterate equation 2 forward by one period, we get

$$V_1(k', A') = \lambda' X_1(k', A')$$

Plugging into the capital FOC, we get

$$\begin{aligned} \lambda &= \beta E\left[\lambda X_1\left(k', A'\right)\right] \\ &= \lambda' \beta E\left[X_1\left(k', A'\right)\right] \\ &= \lambda' \beta E\left[R\left(k', A'\right)\right] \\ \Longrightarrow 1 &= \beta E\left[\frac{\lambda'}{\lambda} R\left(k', A'\right)\right] \end{aligned}$$

Define the stochastic discount factor as

$$M\left(k', A'; k, A\right) \equiv \beta \frac{\lambda'}{\lambda}$$

This is a baby step towards the vast literature of asset pricing. What is the stochastic discount factor? To me, it is a residual term, just like productivity. It captures what we don't quite understand: the shadow forces driving the prices that we do not observe in the data.

1.5 A Lagrangian (Sequential) Approach

Recall the social planner's problem, where we want to maximize the expected future stream of utilities subject to the resource constraints. Let's write it sequentially as,

$$\begin{aligned} V_0 &= \max_{c_t, n_t, k_t, y_t} E_0\left[\sum_{t=0}^{\infty} \beta^t u\left(c_t, n_t\right)\right] \\ \text{s.t. } c_t + k_t &= y_t + (1-\delta) k_{t-1} \\ y_t &= f\left(A_t \frac{n_t}{k_{t-1}}\right) k_{t-1} \end{aligned}$$

Advantages of this setup: Perhaps this setup is more general

Disadvantages: We lose many things, such as existence of solution, concavity, probably computational convenience - this is why in macro literature where solving lifetime utility is warranted, you'll probably see Bellmans much more frequently.

1.5.1 The Lagrangian

Substituting for output,

$$L = \max E_0 \left[\sum_{t=0}^{\infty} \beta^t \left(u\left(c_t, n_t\right) + \lambda_t \left(f\left(A_t \frac{n_t}{k_{t-1}} \right) k_{t-1} + (1-\delta) k_{t-1} - c_t - k_t \right) \right) \right]$$

How to remember which way the sign before the lambda goes: I always use plus, and write the constraint as (income $-$ expenditures). The intuition is, if we have a little more income, how much does our lifetime utility increase? That's why λ in this context is the shadow value or shadow price of wealth.

Key: when we solve the Lagrangian, we are picking our choice variables c_t and n_t based on the information available at date t , not at date 0!

1.5.2 The FOC's

Take the partial differentials wrt the choice variables c_t, n_t and state variables k_t. In the last line, the multiplier λ_t just gives us the budget constraint, so I often omite that.

For simplicity, define

$$\begin{aligned} w_t &\equiv f'\left(A_t \frac{n_t}{k_{t-1}} \right) A_t \frac{1}{k_{t-1}} \\ r_t &\equiv \frac{\partial y_t}{\partial k_{t-1}} = \frac{\partial}{\partial k_{t-1}} \left[f\left(A_t \frac{n_t}{k_{t-1}} \right) k_{t-1} \right] \end{aligned}$$

and moreover define R_{t+1} as the net return

$$\begin{aligned} R_t &\equiv 1 + \text{rate of return} - \text{depreciation} \\ &= 1 + r_t - \delta \end{aligned}$$

$$\begin{aligned}
\frac{\partial L}{\partial c_t} &: \quad u_c(c_t, n_t) = \lambda_t \\
\frac{\partial L}{\partial n_t} &: \quad -u_n(c_t, n_t) = \lambda_t w_t \\
\frac{\partial L}{\partial k_t} &: \quad \lambda_t = \beta E_t[\lambda_{t+1} R_{t+1}] \\
\frac{\partial L}{\partial \lambda_t} &: \quad c_t + k_t = y_t + (1-\delta) k_{t-1}
\end{aligned}$$

You may be wondering, how did we get $\frac{\partial L}{\partial k_t}$? Below I show you:

$$\begin{aligned}
\frac{\partial L}{\partial k_t} &= \beta^{t+1} E_t \left\{ \lambda_{t+1} \left(\frac{\partial}{\partial k_t} \left[f\left(A_{t+1} \frac{n_{t+1}}{k_t} \right) k_t \right] + (1-\delta) \right) \right\} - \beta^t E_t \lambda_t \\
&= \beta^{t+1} E_t \left\{ \lambda_{t+1} \left(\frac{\partial y_{t+1}}{\partial k_t} + (1-\delta) \right) \right\} - \beta^t \lambda_t \\
&= \beta^{t+1} E_t \{ \lambda_{t+1} (r_{t+1} + 1 - \delta) \} - \beta^t \lambda_t \\
&= \beta^{t+1} E_t \{ \lambda_{t+1} (r_{t+1} + 1 - \delta) \} - \beta^t \lambda_t \\
\frac{\partial L}{\partial k_t} &= \beta^{t+1} E_t [\lambda_{t+1} R_{t+1}] - \beta^t \lambda_t = 0 \qquad (4)
\end{aligned}$$

Note: Why is there both λ_{t+1} and λ_t in our partial differential? Remember that λ is a function of t, indeed, the shadow value of wealth may be different every period. Moreover, remember that we are differentiating over a sum, and we have to differentiate every term in the sum. There are TWO terms in this sum that contain k_t,

$$\begin{aligned}
& E_t \left\{ \sum_{t=0}^{\infty} \lambda_t \left(f\left(A_t \frac{n_t}{k_{t-1}} \right) k_{t-1} + (1-\delta) k_{t-1} - c_t - k_t \right) \right\} \\
= \; & E_t \left\{ \cdots + \lambda_t \left(f\left(A_t \frac{n_t}{k_{t-1}} \right) k_{t-1} + (1-\delta) k_{t-1} - c_t - {\color{red}k_t} \right) + \lambda_{t+1} \left(f\left(A_{t+1} \frac{n_{t+1}}{{\color{red}k_t}} \right) {\color{red}k_t} + (1-\delta) {\color{red}k_t} - c_{t+1} - k_{t+1} \right) \ldots \right\}
\end{aligned}$$

When we take the partial derivative, any term not meshed with k_t gets killed.

Note: We have $E_t \lambda_t = \lambda_t$, because E_t is the expectation of λ_t given the information at date t, which is just saying the value of λ_t has already been realized. Thus, there is nothing random left.

Note: Once we rearrange and cancel common terms, we get the usual central asset pricing equation,

$$\begin{aligned} \beta^{t+1} E_t \left[\lambda_{t+1} R_{t+1} \right] - \beta^t \lambda_t &= 0 \\ \lambda_t &= \beta E_t \left[\lambda_{t+1} R_{t+1} \right] \\ 1 &= \beta E_t \left[M_{t+1} R_{t+1} \right] \end{aligned}$$

where M_{t+1} is the stochastic discount factor.

1.6 Decentralization

What do we mean by decentralization? In the social planner's problem above, the planner maximizes utility subject to an economy-wide resource constraint (this constraint is the individual's gross investment minus the consumption). There are three key things one should pay attention to, I think.

- There are no prices paid to use resources in the economy, such as capital and labour.
- The central planner has power over important economic questions:
  - *what* to produce (designer bags or semiconductors)
  - *how much* to produce (should we produce a little or a ton of solar panels)
  - *how* to produce it (should we use human labour or AI machines)
- The goal of the planner is to maximize the utility of consumers.[1]

When we decentralize,

- We introduce **prices** for the resources, commonly denoted as the rate of return on capital r and as the rate of return on labour w.[2]
- The firm has power over important economic questions: what to produce, how much to produce, how to produce it.
- The goal of the firm is to maximize profits.

[1] A side note: we had also implicitly assumed homogeneous consumers, and so maximizing the consumption for one person is really solving the problem for everybody in the economy.

[2] r is often called rental rate in a reference to the history of capital: for much of history, capital was simply land; and thus today, we often call the returns on capital "rent" in reference to the rents people paid on land. Similarly, we often just call w the wages.

1.7 Balanced Growth Path

While we observe in long-run data a fairly constant rate of growth for important macroeconomic variables (consumption, output, investment), often we can simplify our problem by detrending. This means we bring ourselves from an environment of constant positive growth to an environment with zero growth; in other words, we work with the steady state. Mathematically, the steady state corresponds to a set of equations where for all variables x, we have $x_t = x_{t-1} = \bar{x}$, where $\bar{x}$ is the steady state level of x. I primarily see the steady state as a useful tool to study the balanced growth path in a detrended environment.[3]

Recall the equilibrium equations:

$$\begin{aligned}
\text{Feasibility: } c_t + k_t &= y_t + (1-\delta) k_{t-1} \\
\text{Definitions: } y_t &= f\left(A_t \frac{n_t}{k_{t-1}}\right) k_{t-1} \\
r_t k_{t-1} &= y_t - w_t n_t \\
R_t &= r_t + 1 - \delta \\
w_t &= f'\left(A_t \frac{n_t}{k_{t-1}}\right) \frac{A_t}{k_{t-1}} \\
\text{FONCs: } -u_n(c_t, n_t) &= \lambda_t w_t \\
u_c(c_t, n_t) &= \lambda_t \\
\lambda_t &= \beta E_t[\lambda_{t+1} R_{t+1}] \\
\text{Exogenous: } \log A_t &= \rho A_{t-1} + \varepsilon_t, \;\; \varepsilon_t \sim N\left(0, \sigma^2\right) \text{ iid}
\end{aligned}$$

In the steady state,

$$\begin{aligned}
\text{Feasibility: } \bar{c} + \bar{k} &= \bar{y} + (1-\delta) \bar{k} \\
\text{Definitions: } \bar{y} &= f\left(\frac{\bar{n}}{\bar{k}}\right) \bar{k} \\
\bar{r}\bar{k} &= \bar{y} - \bar{w}\bar{n} \\
\bar{w} &= f'\left(\frac{\bar{n}}{\bar{k}}\right) \\
\bar{R} &= \bar{r} + 1 - \delta \\
\text{FONCs: } -u_n(\bar{c}, \bar{n}) &= \bar{\lambda}\bar{w} \\
u_c(\bar{c}, \bar{n}) &= \bar{\lambda} \\
1 &= \beta \bar{R} \\
\text{Exogenous: } \bar{A} &= 1
\end{aligned}$$

[3]Now, some may argue that advanced economies are converging towards a steady state, and some ecologists may argue that this is necessary to preserve the planet, and therefore the steady state may have practical implications. This is an interesting line of inquiry, but I digress.

Practicing solving this system of equations is extremely important!

Key: often in economics we care about relatives instead of absolutes: that is, consumption output ratio $\frac{c}{y}$, capital output ratio $\frac{k}{y}$, or labour and capital shares of income $\frac{wn}{y}$ and $\frac{rk}{y}$, respectively. Two major reasons for the focus on relatives are (1) our research question is about relatives[4] or (2) we cannot identify individual k or y but rather $\frac{k}{y}$.[5]

I demonstrate one way to solve for the ratios:

$$\begin{aligned} 1 &= \beta\bar{R} \implies \bar{R} = \frac{1}{\beta} \\ \bar{r} &= \bar{R} - 1 + \delta \implies \bar{r} = \frac{1}{\beta} - 1 + \delta \end{aligned}$$

Now divide the feasibility constraint and the capital market clearing by the output $\bar{y}$, respectively:

$$\begin{aligned} \frac{\bar{c}}{\bar{y}} + \frac{\bar{k}}{\bar{y}} &= 1 + (1-\delta)\frac{\bar{k}}{\bar{y}} \\ \implies \frac{\bar{c}}{\bar{y}} + \delta\frac{\bar{k}}{\bar{y}} &= 1 \\ \implies \frac{\bar{k}}{\bar{y}} &= \frac{1}{\delta}\left(1 - \frac{\bar{c}}{\bar{y}}\right) \end{aligned}$$

Look at the capital market clearing equation,

$$\bar{r}\frac{\bar{k}}{\bar{y}} = 1 - \frac{\bar{w}\bar{n}}{\bar{y}}$$

Notice that we can sub in for $\bar{r}$ and $\frac{\bar{k}}{\bar{y}}$, so that

$$\left(\frac{1}{\beta} - 1 + \delta\right)\frac{1}{\delta}\left(1 - \frac{\bar{c}}{\bar{y}}\right) = 1 - \frac{\bar{w}\bar{n}}{\bar{y}}$$

Now what is the labour share of income $\frac{\bar{w}\bar{n}}{\bar{y}}$? Recall the labour market clearing condition

$$\bar{w} = f'\left(\frac{\bar{n}}{\bar{k}}\right)$$

[4] How has increasing automation changed the distribution of wealth?

[5] Using relative variables is indispensable when we study misallocation in our research!

Multiplying both sides of the equation by $\bar{n}$ and then dividing by $\bar{y} = f\left(\frac{\bar{n}}{\bar{k}}\right)\bar{k}$, we get that the labour share of income is

$$\begin{aligned}\frac{\bar{w}\bar{n}}{\bar{y}} &= \frac{f'\left(\frac{\bar{n}}{\bar{k}}\right)\bar{n}}{f\left(\frac{\bar{n}}{\bar{k}}\right)\bar{k}} \\ &= \frac{f'\left(\bar{x}\right)}{f\left(\bar{x}\right)}\bar{x}\end{aligned}$$

where $\bar{x} = \frac{\bar{n}}{\bar{k}}$, or the labour capital ratio in steady state.

Moreover, we can divide the labour FOC by the consumption FOC, multiply by $\bar{n}$, divide by $\bar{y}$, to get

$$\begin{aligned}-\frac{u_n\left(\bar{c},\bar{n}\right)}{u_c\left(\bar{c},\bar{n}\right)} &= \frac{\bar{\lambda}\bar{w}}{\bar{\lambda}} = \bar{w} \\ \implies -\frac{u_n\left(\bar{c},\bar{n}\right)\bar{n}}{u_c\left(\bar{c},\bar{n}\right)\bar{y}} &= \frac{\bar{w}\bar{n}}{\bar{y}}\end{aligned}$$

These equations should allow us to solve for the ratios.

1.7.1 Some more important relatives

Define the coeffient of relative risk aversion as

$$\eta = -\frac{u_{cc}\bar{c}}{u_c}$$

Risk aversion in effect measures the curvature of the preference function. To make things "relative", we divide by the slope of the preference and scale by the steady state consumption.

Define the cross-elasticity between consumption and labour as

$$\phi = \frac{u_{cn}\bar{c}}{u_n}$$

The cross-elasticity measures the relative change in consumption when hours worked changes. I interpret dividing by u_n as normalizing out the disutility we derive from work.

Now under certain conditions, η and ϕ are actually related! Recall the King-Plosser-Rebelo 2001 Theorem: that under some mild additional assumptions, the equilibrium equations we wrote above imply that for some $\eta > 0$, some function $\phi(\cdot)$, and up to scaling and constants,

$$u(c,n) = \begin{cases} \frac{(c\phi(n))^{1-\eta}}{1-\eta} & \text{if } \eta \neq 1 \\ \ln(c) + \phi(n) & \text{if } \eta = 1 \end{cases}$$

With this preference specification,

$$\phi = 1 - \eta$$

The intuition is, if we are more risk-averse, we don't want to consume as much per hour worked, because we need to save more. We are afraid of very low consumption in bad times.