Estimating VAR Models with Cholesky and Blanchard Quah Decompositions

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1 Intuition

At this point, we (hopefully) understand why studying impulse responses are useful and how to compute the impulse response. Studying impulse responses help us analyze the effects of different types of shocks (e.g., supply, demand, monetary) on macroeconomic variables like GDP, inflation, and unemployment.

The question is what kinds of shocks should we choose as impulses? Ideally, these shocks are as unrelated as possible to the information we currently have, or "orthogonal". We know something is unrelated to our current information if our prediction is wrong! Yesterday I thought the inflation would decrease, but today I wake up and realize inflation has risen. Since I am wrong, I must be missing information about inflation, so let me go analyze my error. My error is the difference between my forecast at time $t-1$ for time t, and what actually happens at time t. In his slides, Professor Uhlig calls this the one-step ahead prediction error.

In particular, if I am sometimes right and sometimes wrong about inflation, I should look at the variance of my error over time to figure out why I am right and wrong at different times. Call Σ the variance and u_t my prediction error. Let me decompose that variance into orthogonal/unrelated components, so I can isolate the contribution of each of those components that are making me wrong. I should use these components as my impulses to study the path of economic variables! Let me stitch those orthogonal components into a matrix that Professor Uhlig calls A in his slides.

There are different ways I can choose these orthogonal components. Two popular ways are called the Cholesky-decomposition and the Blanchard-Quah decomposition.

2 Cholesky Decomposition

Cholesky Decomposition involves decomposing a positive definite matrix (in our case Σ) into the product of a lower triangular matrix and its transpose. If you have a VAR model involving GDP and inflation, Cholesky Decomposition can help transform the correlated GDP and inflation series into uncorrelated series, facilitating easier modeling and interpretation.

Finding such a decomposition amounts to solving a system of linear equations by assigning unknowns. In the 2x2 case, let

$$A = \begin{pmatrix} a_1 & 0 \\ a_2 & a_3 \end{pmatrix} \text{ s.t. } AA' = \Sigma$$

Suppose $\Sigma = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$ as in Topic 3 Slide 19. Then

$$\begin{aligned} AA' &= \begin{pmatrix} a_1 & 0 \\ a_2 & a_3 \end{pmatrix} \begin{pmatrix} a_1 & a_2 \\ 0 & a_3 \end{pmatrix} \\ &= \begin{pmatrix} a_1^2 & a_1 a_2 \\ a_1 a_2 & a_2^2 + a_3^2 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \end{aligned}$$

So

$$\begin{aligned} a_1^2 = 1 \implies a_1 &= \pm 1 \\ \implies a_2 &= \pm 1 \end{aligned}$$

Notice that a couple solutions are possible: $(a_1, a_2, a_3) = (1, 1, \pm 1), (-1, -1, \pm 1)$. However, by convention (to ensure the uniqueness of the Cholesky decomposition), we restrict ourselves to the case where the diagonal entries of A are real and positive.

Hence the Cholesky matrix is

$$A = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \implies a_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \; a_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

3 Blanchard-Quah Decomposition

Blanchard-Quah Decomposition is used to separate permanent and transitory components of economic shocks. It is based on the idea that not all shocks have a lasting impact on the economy. For example, perhaps the 1917 Russian revolution had a permanent impact on the US economy ;) see https://www.journals.uchicago.edu/doi/10.1086/722933. The Blanchard-Quah decomposition relies on long-run restrictions, where certain variables are assumed not to have a long-term impact on others. For example, a demand shock may not have a long-run effect on output, while a supply shock does.

3.1 First find the transitory component

We want to seek a vector $a = [a_1, a_2]'$ that has no permanent effect on the AR system. Mechanically, that means if a enters as a shock, it eventually dies out. That is, suppose at time 0 $y_0 = a$. Then

$$\begin{aligned} y_1 &= By_0 = Ba \\ \implies y_k &= B^k y_0 \end{aligned}$$

The transitory component must satisfy

$$\lim_{k\to\infty} B^k \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

The problem amounts to just solving the system of equations for a_1 and a_2. The difficulty lies in finding B^k – it would be daunting to multiply it out! Instead, we use a trick. We already know how to decompose $B = VDV^{-1}$ by finding the eigenvalues and eigenvectors. Then

$$\begin{aligned} B &= VDV^{-1} \implies B^k = VD^kV^{-1} \\ \lim_{k\to\infty} B^k a &= V \lim_{k\to\infty} \left(D^k\right) V^{-1} a = 0 \end{aligned}$$

We can easily solve for a after taking $\lim_{k\to\infty}\left(D^k\right)$ and then multiplying out the remaining matrices.

3.2 Next find the Cholesky decomposition and multiply by the inverse

At this point, we could simply pick a vector orthogonal to a and stitch them together to create $\tilde{A}$. But we have more work to do! Why? First, we want $\tilde{A}$ to satisfy $\tilde{A}\tilde{A}' = \Sigma$

because we want the impulse vectors to reflect the contributions of orthogonal components to the error term. Second, we want the vectors in A to be of norm 1. To satisfy the first property, we use the Cholesky decomposition! We exploit the proposition on Topic 3 Slide 13. It turns out, we can employ any Cholesky decomposition to project (or rotate) some vector into the proper impulse response. Suppose we go find the Cholesky decomposition as some matrix $AA' = \Sigma$. Per the proposition (and the proof of the proposition on Slide 14), we multiply the a we found in step 1 by A^{-1}.

$$\tilde{q} = A^{-1}a = \begin{pmatrix} \tilde{q}_1 \\ \tilde{q}_2 \end{pmatrix}$$

3.3 Finally, normalize to length 1

Finally, we want our impulse to be norm 1, so

$$q = \frac{1}{\sqrt{\tilde{q}_1^2 + \tilde{q}_2^2}}\tilde{q} = \begin{pmatrix} q_1 \\ q_2 \end{pmatrix}$$

Pick $q^{\perp} = \begin{pmatrix} -q_2 \\ q_1 \end{pmatrix}$, which is an orthogonal vector to q. Hence,

$$A^{BQ} = AQ = A \begin{pmatrix} q_1 & -q_2 \\ q_2 & q_1 \end{pmatrix}$$

is the Blanchard-Quah decomposition of Σ, where A is the Cholesky decomposition.